On the Fundamental Group of a Certain Class of Plane Algebraic Curves

A Dissertation

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

W. S. TURPIN

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R. UNIV.
BIBLIOTHEEK
LEIDEN

ON THE FUNDAMENTAL GROUP OF A CERTAIN CLASS OF PLANE ALGEBRAIC CURVES.*

By W. S. Turpin.

1. Introduction. The problem of existence of algebraic functions, z, of two independent variables, x and y, possessing a preassigned branch curve of order n

$$f_n(x, y) = 0 \tag{1}$$

has been considered by Enriques [1] and Zariski.[2] Zariski has shown that, in view of a result of Enriques, this question may be reduced to the consideration of the Poincaré (fundamental) group of the residual space of the branch curve (1) relative to its carrying complex projective plane and the application of the Riemann existence theorem for algebraic functions of one variable having preassigned branch points.

It is sufficient for the theory of algebraic surfaces from the point of view of birational transformations to consider branch curves (1) possessing only ordinary double points and cusps. Zariski has shown that if a curve possesses only ordinary double points then its fundamental group is necessarily cyclic. A simple case of a curve whose fundamental group is not cyclic is that of the branch curve of a cubic surface. If the cubic surface is general, its branch curve is a sextic, f_6, with six cusps on a conic:

$$f_6 : [\phi_3(x, y)]^2 + [\psi_2(x, y)]^3 = 0, \tag{2}$$

where ϕ_3 and ψ_2 are polynomials in x and y of respective degrees 3 and 2. This curve was treated in detail by Zariski and its fundamental group specifically determined.

An obvious generalization of the curve (2) is the curve f_{6m}, of order $6m$, with $6m$ cusps at the intersections of two curves, $\phi_{3m}(x, y) = 0$ and $\psi_{2m}(x, y) = 0$, of orders $3m$ and $2m$ respectively, where m is a positive integer. Such a curve is given by the equation:

$$f_{6m} : [\phi_{3m}(x, y)]^2 + [\psi_{2m}(x, y)]^3 = 0. \tag{3}$$

* Received December 1, 1936.

[1] F. Enriques, "Sulla costruzione delle funzioni algebriche di due variabili possedenti una data curva di diramazione," *Annali di matematica pura ed applicata*, ser. 4, vol. 1 (Nov., 1923), pp. 185-198.

[2] O. Zariski, "On the problem of the existence of algebraic functions," *American Journal of Mathematics*, vol. 51 (1929), pp. 305-328.

The methods of investigation used for the fundamental group of (2) are peculiar to the sextic of this type and do not admit of an extension to the more general class of curves (3). Hence, it was deemed of interest to investigate the structure of the fundamental group of curves of the type (3). One may expect that the methods developed in this investigation may point the way to a possible procedure for other types of curves, for instance for the branch curves of general surfaces of any order in S_3. As it is known, these curves have been completely characterized by B. Segre.[3]

This investigation of the structure of the fundamental group falls under three classifications:

1°. The determination of the fundamental group, G, of a degenerate limit curve, $\bar{f}$, of curves (3).

2°. The factorization of the relations of $\bar{G}$ into relations belonging formally to the fundamental group G' of a virtual curve f' with $6m^2$ cusps, of which $\bar{f}$ is a limit curve.

3°. Verification of the fact that $G' = G$, where G denotes the fundamental group of a curve f of type (3).

Our method of attacking the problem in our special case contains the nucleus of a perfectly general procedure, applicable to an arbitrary plane curve with nodes and cusps, provided the complete continuous (irreducible) system $\{f\}$ of curves having the same singularities as f contains some special curve $\bar{f}$, for instance a degenerate curve without multiple components, whose fundamental group can be directly determined. However, to date, a method has not been found for the step 3° of the above procedure that does not appeal to the special geometry of the curves f. This verification is necessary due to the fact that the factorization obtained in 2° is not unique. It seems probable that an equivalence of possible factorizations can be established by purely group theoretic considerations, but, efforts in this direction have not been successful up to the present.

2. General properties of the fundamental group G and its associated group T.[4] Consider the curve f determined by the equation $f(x, y) = 0$ where $f(x, y)$ is a polynomial, of degree n, in the complex variables x and y.

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[3] B. Segre, "Sulla caratterizzazione delle curve di diramazione nei piani multipli generali," *Mem. Accad. Ital.*, Mat., vol. 1 (1930).

[4] The concepts and results in this section are compiled from the following sources: S. Lefschetz, "Topology," *American Mathematical Society Colloquium Publications*, vol. 12 (1930); E. R. van Kampen, "On the fundamental group of an algebraic curve," *American Journal of Mathematics*, vol. 55 (1933); O. Veblen, "Analysis situs," *American Mathematical Society Colloquium Publications*, vol. 5 (1922); O. Zariski, "On the

We shall be interested in the Poincaré group, G, of the residual space, S, of the curve f relative to its carrying complex projective plane (x, y).

Let us suppose that the coördinate axes have been chosen in such a manner that f does not pass through the point at infinity on the y-axis and that it possesses no multiple components. A generic line of the pencil $\{x = \text{const.}\}$ will thus have n distinct intersections with the curve f. Moreover, lines of this pencil having less than n distinct intersections with f are finite in number. We shall call such lines singular lines of the pencil and denote them by $x = \alpha_i$ $(i = 1, 2, \cdots, \nu)$. Denote by α_0 a point distinct from the set $[\alpha_i]_{i=1, 2, \ldots, \nu}$ and let $[\delta'_i]_{i=1, 2, \ldots, \nu}$ be a set of non-intersecting loops in the plane of the variable x emanating from $x = \alpha_0$ and surrounding respectively the points of the set $[\alpha_i]_{i=1, 2, \ldots, \nu}$. Let $y_k = b_k$ $(k = 1, 2, \cdots, n)$ be the roots of $f(\alpha_0, y) = 0$ and choose loops g_k in the plane of the complex variable y, emanating from the point at infinity and surrounding b_k, in such a manner that g_i and g_j have only the point at infinity in common for $i \neq j$. We denote this base point of the loops g_k by O.

If we define the Poincaré group G of f in the usual manner, it is well known that the loops g_k may be taken as generators of G and that G is independent of the choice of O to within an isomorphism. It is evident that the generators g_k satisfy the following relation:

$$g_1 g_2 \cdots g_n = 1. \tag{4}$$

As x traverses a closed path in its plane, starting from and returning to α_0, the set of points $[b_k]$ will move continuously describing certain paths in the y: plane and returning finally to its original position, although the individual points may have been permuted among themselves. As the points b_k move, their corresponding loops g_k will also vary and this variation is completely determined by the motion of the points b_k and the condition that the set $[g_k]$ should always consist of non-intersecting loops.

If, under cyclical variation of x from α_0, a root y traverses a path from b_i to b_j, the loop g_i is transformed into a loop g'_i surrounding b_j alone and which must therefore be a transform of g_j by some element of G. Moreover, g_i and g'_i are equivalent elements of G and, thus, corresponding to every cyclical variation of x from α_0, we obtain a relation between the elements of G. In particular, if x traverses the loops $[\delta'_i]_{i=1, \ldots, \nu}$ we obtain motions of the roots y_k which yield the relations

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problem of existence of algebraic functions," *loc. cit.*; O. Zariski, "On the Poincaré group of rational plane curves." *American Journal of Mathematics*, vol. 58 (1936).

$$(5) \qquad \Phi_{k,i}(g_1, g_2, \cdots, g_n) \equiv g_k^{-1} g_{k,i}^{-1} g_{p_{ki}} g_{ki} = 1 \qquad \begin{pmatrix} k = 1, 2, \cdots, n \\ i = 1, 2, \cdots, \nu \end{pmatrix}$$

where $\{p_{1,i}, p_{2,i}, \cdots, p_{n,i}\}$ is a permutation of $\{1, 2, \cdots, n\}$. It has been shown that the relations (5) together with the relation (4) constitute a complete set of generating relations for the group G.

The class of motions of the set of points $[b_k]$ induced by cyclic variation of x incident to the point α_0 which carry this set into its original position,

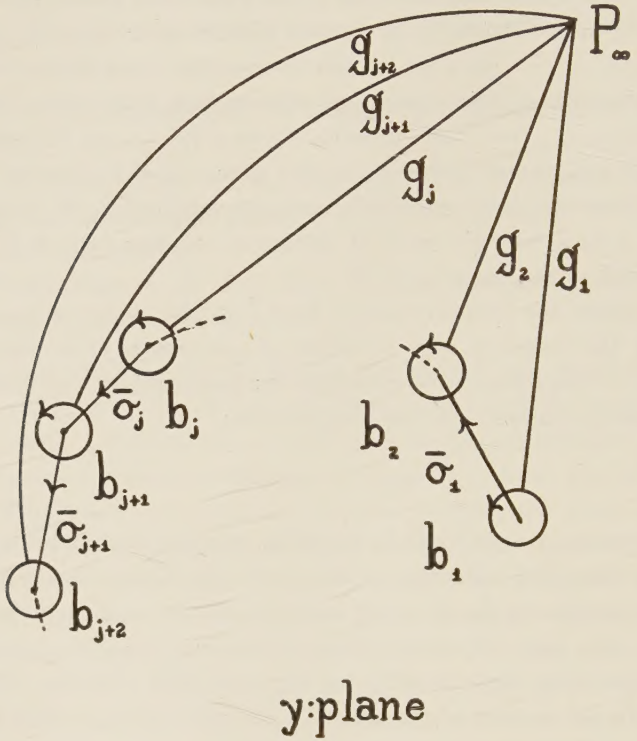

Fig. 1.

in such a way that the roots y_k remain distinct during the motion, constitutes a group of motions T. Two such motions, m_1 and m_2, correspond to the same element of T if the motion m_1 may be deformed into the motion m_2 by suitable deformation of the path of x so that, during the latter deformation, the induced motions keep the roots y_k distinct.

Let $\bar{\sigma}_j$ $(j = 1, 2, \cdots, n-1)$ denote an oriented arc from b_j to b_{j+1} of such a type that the adjacent arcs have only an end point in common and non-adjacent arcs do not intersect (see Fig. 1). Let T_j $(j = 1, 2, \cdots, n-1)$

denote a motion in which the points b_i $(i \neq j, j+1)$ are fixed while the points b_j, b_{j+1} are interchanged, y_j moving from b_j to b_{j+1} along the right-hand edge of $\bar{\sigma}_j$ and y_{j+1} moving from b_{j+1} to b_j along the opposite edge. Any motion, m, belonging to T can be deformed into a motion, m', which may be expressed as a product of the elementary motions T_j. The deformation of m into m' affects not only the paths but also the velocities of the points b_k. The elementary motions T_j satisfy the relations:

$$T_i T_j = T_j T_i, \qquad |i-j| \neq 1 \tag{6}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}. \tag{7}$$

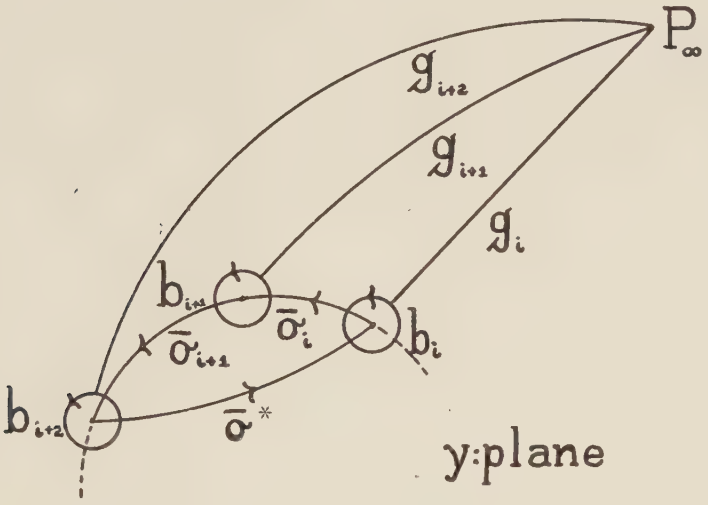

Fig. 2.

The commutative relation, (6), holds due to the fact that, under the assumption $|i-j| \neq 1$, $\bar{\sigma}_i$ and $\bar{\sigma}_j$ have nothing in common and the motions T_i and T_j are thus independent of the order of their performance. The proof of (7) is as follows: Introduce the motion T^* sending y_{i+2} from b_{i+2} to b_i along the right-hand edge of the oriented arc $\bar{\sigma}^*$ and y_i from b_i to b_{i+2} along its opposite edge, where $\bar{\sigma}^*$ is chosen as indicated in Fig. 2. Then the following equalities hold; $T^* T_{i+1} = T_{i+1} T_i = T_i T^*$. Solving these for T^* and equating the solutions, we have that $T_{i+1} T_i T_{i+1}^{-1} = T_i^{-1} T_{i+1} T_i$. Multiplying this relation on the right by T_{i+1} and on the left by T_i, we obtain the desired result.

If we now choose g_1 to be any loop on O surrounding b_1 which does not intersect the arcs $\bar{\sigma}_2, \cdots, \bar{\sigma}_n$, and let g_i be the loop into which g_{i-1} is deformed as y_{i-1} moves from b_{i-1} to b_i along the arc $\bar{\sigma}_{i-1}$ for $(i = 2, \cdots, n)$, then the

motion T_j induces a transformation of the following type on the set of loops $[g_k]$:

$$(8) \qquad t_j : \begin{cases} g'_i = g_i, & (i \neq j, j+1) \\ g'_j = g_{j+1} \\ g'_{j+1} = g_{j+1}^{-1} g_j g_{j+1} & (j = 1, 2, \cdots, n-1). \end{cases}$$

If we denote the complement of the set of points $[\alpha_i]_{i=1, 2, \ldots, \nu}$ relative to the plane of the complex variable x by $C[\alpha_i]$, then, as x describes a cyclic path incident to α_0 in $C[\alpha_i]$, the points $[b_k]$ undergo a motion belonging to the group T, the loops g_k are subjected to the corresponding transformation which, in turn, yields a generating relation of the group G. In particular, the motions $T_{\delta'_i}$ generated as x describes the loops δ'_i induce transformations which yield the relations (5) for G. In consequence, a knowledge of the motions $T_{\delta'_i}$ is sufficient to determine the structure of G.

It will be useful to examine the motion induced on the roots y as x describes a loop about a singular value corresponding respectively to a tangent, ordinary double point and cusp of f.

(A) Suppose $x = \alpha_i$ is a simple tangent to the curve f and that $y_1 = b_1$, $y_2 = b_2$ are the two roots of $f(\alpha_0, y) = 0$ which tend towards coincidence as $x \to \alpha_i$. Then, as x describes δ'_i, the motion of the roots y_k is equivalent to one in which y_j is fixed for $j > 2$ and the motion of y_1 and y_2 is typified by that of the roots of $y^2 = x$ as x describes the loop $| x | = 1$. This motion has the form T_1. If we do not make the simplifying assumption to the effect that the two roots which approach coincidence as $x \to \alpha_i$ have consecutive indices, the corresponding motion will have the form $\bar{T}^{-1} T_1 \bar{T}$ where $\bar{T}$ is a product of elementary motions T_j.

(B) Suppose $x = \alpha_i$ is a singular value corresponding to an orrdinary double point of f. Then the motion of the roots induced as x describes δ'_i is equivalent to T_1^2, or, in the general case, to $\bar{T}^{-1} T_1^2 \bar{T}$. Such a singular value α_i may be considered as the limit of two singular values α_i' and α_i'', corresponding to simple tangents of f, at each of which the same pair of roots is permuted.

(C) Suppose $x = \alpha_i$ is a singular value corresponding to a cusp of f. Then the motion of the roots induced as x describes δ'_i is equivalent to T_1^3, or, in the general case, to $\bar{T}^{-1} T_1^3 \bar{T}$. In this case, the singular value α_i can be considered as the limit of three singular values corresponding to simple tangents which have approached coincidence and at all of which the same pair of roots is permuted.

The relations arising between the generators g_i of G due to singular values of types (A), (B) and (C) are, respectively, of the following forms:

(a) $g_1 = g_2$ (b) $g_1 g_2 = g_2 g_1$ (c) $g_1 g_2 g_1 = g_2 g_1 g_2$.

Let us now consider a variable curve f and let $\bar{f}$ denote a limit curve under continuous variation of f. If f and $\bar{f}$ have the same singularities, they are isotopic and, accordingly, possess the same fundamental group. Suppose, however, that as f tends towards $\bar{f}$ it acquires new multiple points or multiple points of higher order. For the sake of simplicity and definiteness, let us suppose that as $f \to \bar{f}$ the simple singular values α_1 and α_2 of f approach coincidence in α so that $\bar{f}$ acquires a new ordinary double point. Any generating relation for f corresponding to a singular value distinct from α_1, α_2 remains a true relation for $\bar{f}$. The relations for f relative to α_1 and α_2, viz. $g_1 = g_2$, are destroyed for $\bar{f}$ since, in the limit when $\alpha_1, \alpha_2 \to \alpha$, both the loops δ'_1 and δ'_2 pass through α. On the other hand, the relation $g_1 g_2 = g_2 g_1$ for f which arises from a circuit, δ', of x about both α_1 and α_2 is not destroyed for $\bar{f}$ since, in the limit, it becomes the relation corresponding to a circuit of x about the singular value α for $\bar{f}$. This reasoning is perfectly general and applies to any new multiple point, or point of higher order, acquired by $\bar{f}$. In fact, the following theorem has been established:

All generating relations of the fundamental group $\bar{G}$ of a limit curve $\bar{f}$ of a variable curve f are also true generating relations for the fundamental group G of f. This theorem holds even if $\bar{f}$ is degenerate provided that it possesses no multiple components.

3. Consideration of the motions, T, occurring at the singular values of a degenerate case of $f_{6m}(x, y)$. We wish to investigate the fundamental group of curves of the type

$$f_{6m}(x, y) \equiv [\phi_{3m}(x, y)]^2 + [\psi_{2m}(x, y)]^3 = 0 \tag{3}$$

where ϕ_{3m} and ψ_{2m} are polynomials in the two complex variables x and y of respective degrees $3m$ and $2m$. It is supposed that the intersections of $\phi_{3m} = 0$ and $\psi_{2m} = 0$ are distinct. If the curves $\phi_{3m} = 0$ and $\psi_{2m} = 0$ are general, the curve $f_{6m} = 0$ will possess $6m^2$ cusps at the intersections of the curves $\phi_{3m} = 0$ and $\psi_{2m} = 0$ and no further singular points. For a general choice of coördinate axes, f_{6m} will possess, under these restrictions, $6m^2$ distinct critical values corresponding to cusps and $6m(3m - 1)$ distinct critical values corresponding to tangents.

We postpone the consideration of (3) temporarily and consider a degenerate case of a curve of this type, namely,

$$\bar{f}'_{6m}(x, y) \equiv y^{6m} - x^{3m} = 0. \tag{9}$$

The critical values for this function are $x = 0$ and $x = \infty$. The singularity corresponding to each of these values consists of $3m$ branches having simple contact and vertical branch tangent. We now proceed to ascertain the motion

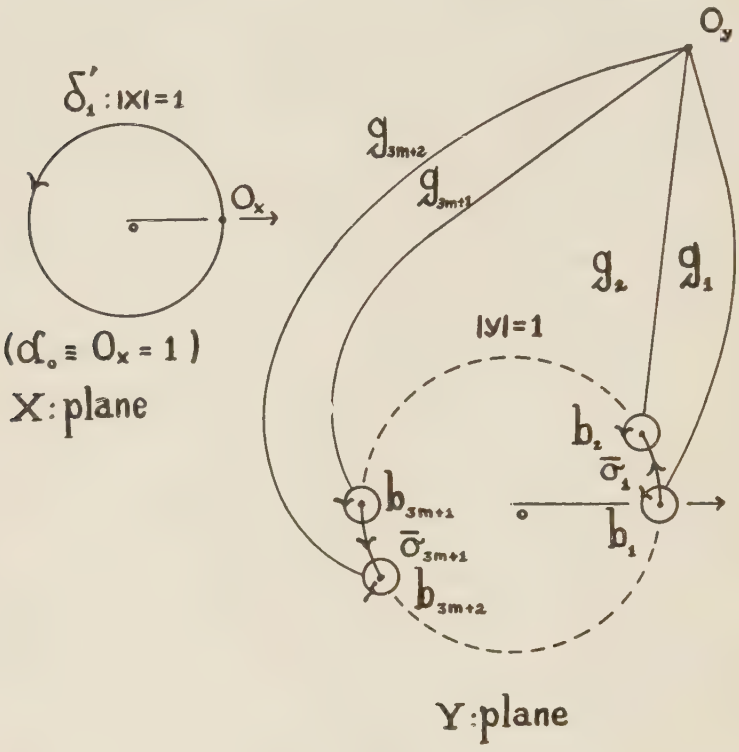

Fig. 3.

T' induced as x describes a loop about the critical value $x = 0$. Let us choose $x = \alpha_0$ to be $x = 1$ and as δ'_1 choose $| x | = 1$. Now, the roots of $f'_{6m}(0, y) = 0$, which we denote by b_k, are the $6m$-th roots of unity. Let us select loops g_i in the y: plane as indicated in Fig. 3. Then as x describes the loop δ'_1, a motion T' is set up which sends b_i into the diametrically opposite point b_{3m+i} along the arc $\overrightarrow{b_i b_{3m+i}}$. Let T_i denote the elementary motion corresponding to the arc $\bar{\sigma}_i$ of Fig. 3. Then the motion T' may be expressed in terms of the elementary motions T_i as follows:

$$T' = (T_{6m-1}T_{6m-2}\cdots T_2T_1)^{3m}. \tag{10}$$

Since the loop δ'_1 may also be considered as a loop about $x = \infty$, the motion of the roots induced as x describes a loop about the latter critical point is again equivalent to T'.

4. Factorization of the motion T'^2 into elementary motions belonging to virtual cusps and simple tangents. The curve (9) is a member of the continuous subsystem of the system $\{\bar{f}_{6m}\}$ consisting of the curves

$$\bar{f}^*_{6m} : y^{6m} = (x-a)^{3m}(x-b)^{3m}, \tag{11}$$

and is isotopic to the general curve of this subsystem. In fact, the equations

$$x' = \frac{x - \tau b}{\tau x + 1 - \tau(1+a)}, \qquad y' = \frac{y}{\tau x + 1 - \tau(1+a)} \tag{12}$$

where τ is a parameter, define for each value of τ such that

$$\tau b - \tau(1+a) + 1 \neq 0,$$

a non-degenerate collineation π_τ in the projective (x, y) plane which carries the curve (9) into a curve of the system $\{\bar{f}^*_{6m}\}$. Moreover, π_0 is the identity while $\pi_1(\bar{f}^*_{6m})$ is the curve (9).

The curve $\bar{f}^*_{6m}$ is a limit curve of the general curve, f_{6m}, and we have, in the pencil $\{x = \text{const.}\}$, only two singular lines for $\bar{f}^*_{6m}$, $x = a$ and $x = b$. These singular lines each absorb a certain number of singular lines with respect to the general curve, f_{6m}, and since the singular lines $x = a$ and $x = b$ can be interchanged by a continuous variation of the curve $\bar{f}^*_{6m}$ in the system (11), it follows that each of these lines absorbs $3m^2$ lines $x = \text{const.}$ passing through cusps of f_{6m} and $3m(3m-1)$ simple tangents of f_{6m}. It must therefore be possible to factor the motion T'^2 into a product of motions $\bar{T}_i$, $6m^2$ of which correspond to cusps and the remaining $6m(3m-1)$ of which correspond to simple tangents. We proceed to exhibit formally one such factorization of T'^2, making use of the relations (6) and (7), and we shall show afterwards that this factorization actually belongs to the curve $\bar{f}^*_{6m}$ considered as a limiting case of the general curve of the system $\{f_{6m}\}$.

We first write T'^2 in the form

$$T'^2 = (T_{6m-1}T_{6m-2}\cdots T_1)^{6m} = [(T_{6m-1}T_{6m-2}\cdots T_1)^2]^{3m}. \tag{13}$$

It will be useful to establish two lemmas.

LEMMA 1. $T_jT_{j-1}T_{j-2}(T_jT_{j-1})^{-1} = (T_{j-1}T_{j-2})^{-1}T_jT_{j-1}T_{j-2}$.

Proof. Making use of an alternate form of (7) we have

$$T_jT_{j-1}T_{j-2}T_{j-1}^{-1}T_j^{-1} = T_jT_{j-2}^{-1}T_{j-1}T_{j-2}T_j^{-1}$$

and in virtue of (6)

$$T_jT_{j-2}^{-1}T_{j-1}T_{j-2}T_j^{-1} = T_{j-2}^{-1}T_jT_{j-1}T_j^{-1}T_{j-2}.$$

Repeating the first process, we obtain

$$T_{j-2}^{-1}T_jT_{j-1}T_j^{-1}T_{j-2} = T_{j-2}^{-1}T_{j-1}^{-1}T_jT_{j-1}T_{j-2}$$

which is the desired result.

LEMMA 2. $(T_jT_{j-1})^2 = (T_j^{-1}T_{j-1}T_j)T^3{}_{j-1}$.

Proof. Using (7), we may write

$$T_jT_{j-1}T_jT_{j-1} = T_{j-1}T_jT^2{}_{j-1} = (T_{j-1}T_jT_{j-1}^{-1})T^3{}_{j-1}$$

and making use of an alternate form of (7) we are enabled to write the last member in the desired form.

Let us now write $(T_{6m-1}T_{6m-2}\cdots T_1)^2$ in the following manner:

$$\begin{aligned}\{[T_{6m-1}\cdots T_1][(T_{6m-1}T_{6m-2})^{-1}(T_{6m-4}T_{6m-5})^{-1}&\\ \cdots(T_2T_1)^{-1}(T_{6m-1}T_{6m-2})^2(T_{6m-4}T_{6m-5})^2&\\ \cdots(T_2T_1)^2(T_{6m-1}T_{6m-2})^{-1}(T_{6m-4}T_{6m-5})^{-1}&\\ \cdots(T_2T_1)^{-1}][T_{6m-1}\cdots T_1]\}.&\end{aligned}$$

This arrangement is possible due to the fact that the center bracket reduces to the identity in virtue of the relation (6). If we again make use of (6), this expression may be rearranged as follows:

$$\begin{aligned}\{[T_{6m-1}T_{6m-2}T_{6m-3}\cdot(T_{6m-1}T_{6m-2})^{-1}T_{6m-4}T_{6m-5}T_{6m-6}\cdot(T_{6m-4}T_{6m-5})^{-1}&\\ \cdots T_5T_4T_3\cdot(T_5T_4)^{-1}T_2T_1(T_2T_1)^{-1}][(T_{6m-1}T_{6m-2})^2(T_{6m-4}T_{6m-5})^2&\\ \cdots(T_2T_1)^2][(T_{6m-1}T_{6m-2})^{-1}T_{6m-1}T_{6m-2}(T_{6m-4}T_{6m-5})^{-1}\cdot T_{6m-3}T_{6m-4}T_{6m-5}&\\ \cdots(T_2T_1)^{-1}\cdot T_3T_2T_1]\}.&\end{aligned}$$

If we perform the obvious cancellations and apply Lemma 1 to the last bracket, we obtain:

$$\begin{aligned}\{[T_{6m-1}T_{6m-2}T_{6m-3}(T_{6m-1}T_{6m-2})^{-1}\cdots T_5T_4T_3(T_5T_4)^{-1}][(T_{6m-1}T_{6m-2})^2&\\ \cdots(T_2T_1)^2]T_{6m-3}T_{6m-4}T_{6m-5}(T_{6m-3}T_{6m-4})^{-1}\cdots T_3T_2T_1(T_3T_2)^{-1}]\},&\end{aligned}$$

and, applying Lemma 2 to the elements of the middle bracket, we are thus enabled to write T'^2 in the form

$$(14)\quad T'^2 = \{[T_{6m-1}T_{6m-2}T_{6m-3}(T_{6m-1}T_{6m-2})^{-1}\cdots T_5T_4T_3(T_5T_4)^{-1}]$$
$$[(T^{-1}_{6m-1}T_{6m-2}T_{6m-1})T^3{}_{6m-2}(T^{-1}_{6m-4}T_{6m-5}T_{6m-4})T^3{}_{6m-5}\cdots(T_2^{-1}T_1T_2)T_1^3]$$
$$[T_{6m-3}T_{6m-4}T_{6m-5}(T_{6m-3}T_{6m-4})^{-1}\cdots T_3T_2T_1(T_3T_2)^{-1}]\}^{3m},$$

which is the desired type of factorization.

5. A function f^*_{6m}, of type (3), whose critical values correspond to induced motions given by the factors of (14). Consider the curve

$$(15)\qquad F_{6m}(x,y) \equiv (x^{3m}-1)^2 - (y^{2m}-1)^3 = 0.$$

The critical values of the function $F_{6m}(x,y)$ are divided into two classes according to the following classification:

1. $x = a_j = \omega^j{}_{3m}$ $(j = 0, 1, \cdots, 3m-1)$, where $\omega^j{}_{3m}$ denote the $3m$-th roots of unity. Each of these critical values corresponds to a line containing $2m$ cusps of the curve $F_{6m} = 0$, each having a vertical cusp tangent.
2. $x = c_{k,l} = |\,2\,|^{1/6m}\cdot e\left[i\pi\dfrac{(-1)^k+8l}{12m}\right]$ for $k = 0, 1$ and $l = 0, 1, \cdots, 3m-1$. Each of these values corresponds to a flex tangent having contact of order $2m-1$ with the curve $F_{6m} = 0$.

Let us choose $x = 0$ as common origin of loops δ'_j, δ'_{kl} in the x: plane selected as indicated in Fig. 4. The roots $y_k = b_k$ of $F_{6m}(0,y) = 0$ are given by:

$$b_{1+3j} = e\left[i\pi\frac{(6j-1)}{6m}\right]$$
$$b_{2+3j} = |\,2\,|^{1/2m}\cdot e^{i\pi j/m}$$
$$b_{3+3j} = e\left[i\pi\frac{(6j+1)}{6m}\right]$$

for $j = 0, 1, \cdots, 2m-1$. We now choose oriented arcs $\overline{\sigma}_k$ as indicated in the diagram of the y: plane of Fig. 4 and choose loops g_k, surrounding b_k, in the manner outlined in Sec. 4. Let us examine the motion induced on the roots y_k as x describes one of the loops δ' in the x: plane. This examination will be somewhat simplified if we also consider the auxiliary curve

$$(16)\qquad Y^3 = X^2$$

obtained from the curve $F_{6m}(x,y) = 0$ by setting $Y = y^{2m}-1$ and $X = x^{3m}-1$. The critical values of the function $F_{6m}(x,y)$ are evidently divided into three classes: the values a_j corresponding to the cusp of (16), the values $c_{0,l}$ corresponding to the ordinary value $X = i$ and the values $c_{1,l}$ corresponding to the

ordinary value $X = -i$. The initial value, $x = 0$, of the loops δ' corresponds to the ordinary value $X = 1$. It is therefore clear that as x describes a loop δ'_j, X describes a loop Δ, and, as x describes a loop δ'_{kl}, X describes a loop Δ_k where Δ, Δ_0, Δ_1 are loops in the X: plane emanating from $X = 1$ and surrounding respectively the values $X = 0, i, -i$. If we denote the roots Y_n of (16) for $X = 1$ by B_n where $B_n = e[(2\pi i/3)(n-2)]$ $(n = 1, 2, 3)$, then, clearly, the image of the points b_{n+3j} in the Y: plane is the point B_n for all values of j. Let us define elementary motions T_i of the points b_k with reference

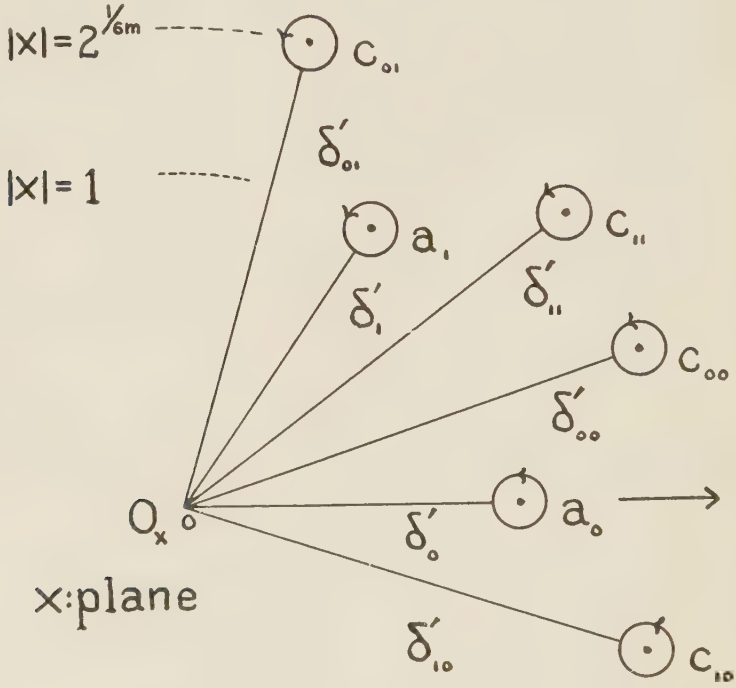

Fig. 4x.

to the oriented arcs $\bar{\sigma}_k$ of Fig. 4, and elementary motions T^*_i of the points B_k with reference to arcs $\bar{\Sigma}_i$ chosen as positively oriented arcs of the unit circle joining B_i to B_{i+1} for $i = 1, 2$. As x describes the loop δ'_s, X describes the loop Δ and, therefore, the points B_n undergo the motion $(T^*_2 T^*_1)^2$. It therefore follows that the image points, b_{n+3j}, in the y: plane are subjected to the motion $(T_{2+3j} T_{1+3j})^2$ for all values of j. Thus, as x describes the loop δ'_s, we have that the corresponding motion of the points b_k is given by

$$(T_2 T_1)^2 (T_5 T_4)^2 \cdots (T_{6m-1} T_{6m-2})^2 \tag{17}$$

for $s = 0, 1, \cdots, 2m - 1$.

Suppose x describes the loop δ'_{ks}; then X describes the loop Δ_k. Since Δ_k does not surround a critical value of the function (16), it follows that the motion T^* of the points B_n is the identity. Consequently, the motion of the points b_k must leave the set $[b_{n+3j}]_{(j)}$ invariant for each n. This motion is due to the flex line $x = c_{k,s}$ of the pencil $\{x = \text{const.}\}$ and interchanges the points $b_{(3-2k)+3j}$ in a cyclical manner. This motion has the following description in terms of the elementary motions T_i:

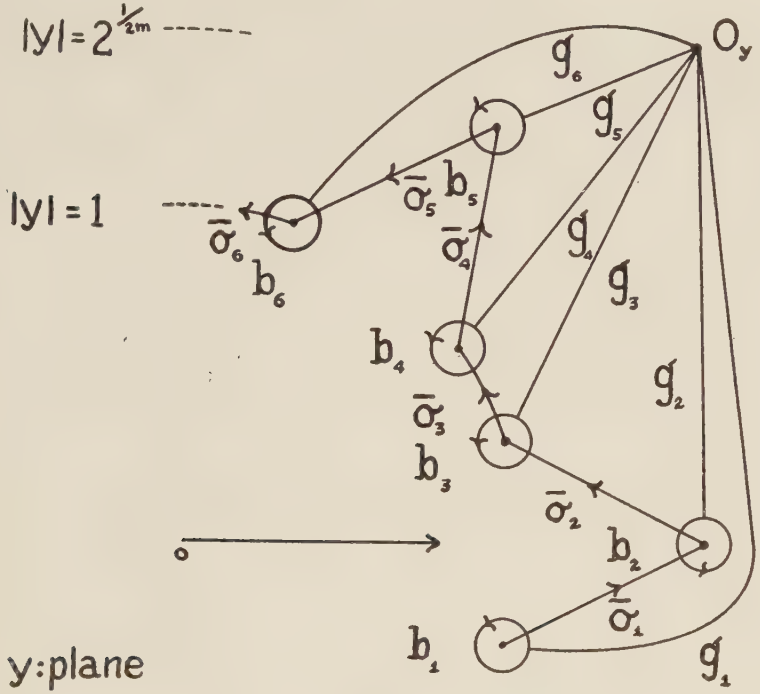

Fig. 4y.

$$(18)\qquad T^{(k)} = \prod_{j=0}^{2m-2}{}_R \left(T_{6m+2k-3j-3}T_{6m+2k-3j-4}T_{6m+2k-3j-5}(T_{6m+2k-3j-3}T_{6m+2k-3j-4})^{-1}\right)$$

where $\prod_R$ indicates multiplication on the right.

For the curve $F_{6m}(x, y) = 0$, certain of the singular lines of the pencil $\{x = \text{const.}\}$ are coincident. These coincidences are not intrinsic and are due to the special choice of the coördinate axes. In order to eliminate these coincidences from the considerations, we proceed in the following manner.

If we denote by $F_{6m}(\theta)$ the function obtained by rotating F_{6m} through a positive angle θ, then, for small values of θ, the curve $F_{6m}(\theta) = 0$ will have cusps near to the cusps of $F_{6m} = 0$; however, the cusp tangents will no longer

be vertical. In fact, the critical value $x = a_j$ of F_{6m} has associated with it critical values $x = a_{jk}$ corrresponding to cusps of $F_{6m}(\theta) = 0$ and values $x = \bar{a}_{jk}$ corresponding to simple tangents of the curve $F_{6m}(\theta) = 0$, where a_{jk} and $\bar{a}_{jk}$ are in the vicinity of a_j, for $k = 1, 2, \cdots, 2m$.

When a singular line of the pencil $\{x = \text{const.}\}$, which passes through a cusp having this singular line as cusp tangent, breaks up into a pair of singular lines, one of which is a simple tangent and the other a line through the cusp, the motion $(T_{s+1}T_s)^2$ breaks up into either the product of ${T^3}_{s+1}$ and $T^{-1}_{s+1}T_sT_{s+1}$ or of ${T_s}^3$ and $T^{-1}_{s+1}T_sT_{s+1}$, according as which of the two, essentially distinct, possible choices of loops in the x: plane is selected. If we are interested merely in the generating relations of the fundamental group and not in the actual motion of the roots, it is indifferent which of these choices is made, due to the fact that the resulting relations among the generators of the fundamental group are the same in both cases.

The singular lines of the pencil $\{x = \text{const.}\}$ passing through flexes of $F_{6m} = 0$ break up into distinct singular lines passing through points of simple tangency of $F_{6m}(\theta) = 0$. Once more, there is ambiguity concerning the determination of the actual motions corresponding to these singular lines, the motions again being dependent on the choice of loops. However, the flex as a unit imposes the same relations on the generators g_{ik} as does the corresponding number of distinct simple tangents which approach coincidence to form the flex. Consequently, for any choice of loops, the motions corresponding to these simple tangents must impose the same relations on the elements g_k as the factors of the flex motion (18) which correspond to simple tangents. Thus, from the standpoint of relations on the generators g_k, it is sufficient to treat the set of singular lines which approach coincidence in a flex tangent as a unit and merely say that the imposed relations are those of the original flex, the individual motions corresponding to the tangents being left out of the consideration entirely.

To summarize, $F_{6m}(\theta)$ will have the following properties:

1°. Possesses $6m^2$ critical values a_{jk} corresponding to cusps of the curve $F_{6m}(\theta) = 0$ such that, for a proper choice of loops in the x: plane surrounding these values, the induced motions are $(T_{3k+1})^3$ where $k = 0, 1, \cdots, 2m - 1$ and $j = 0, 1, \cdots, 3m - 1$.

2°. Possesses $6m^2$ critical values $\bar{a}_{jk}$ corresponding to simple tangents of the curve $F_{6m}(\theta) = 0$, such that, for a proper choice of loops in the x: plane, the induced motions are $T^{-1}_{3k+2}T_{3k+1}T_{3k+2}$ for $k = 0, 1, \cdots, 2m - 1$ and $j = 0, 1, \cdots, 3m - 1$.

3°. Possesses $6m$ sets of $2m - 1$ critical values c_{klt} corresponding to

simple tangents of the curve $F_{6m}(\theta) = 0$ of such a type that, for a proper choice of loops in the x: plane, the induced motions relative to a set c_{klt} for k and l fixed impose the same relations on the generators g_j as do the motions

$$(19) \qquad T_{6m+2k-3t-3}T_{6m+2k-3t-4}T_{6m+2k-3t-5}(T_{6m+2k-3t-3}T_{6m+2k-3t-4})^{-1}$$

for $t = 0, 1, \cdots, 2m-2$, where k and l may assume any of the values $k = 0, 1$; $l = 0, 1, \cdots, 3m-1$.

These properties are exactly those desired for the function f^*_{6m} and, accordingly, we define f^*_{6m} to be the function $F_{6m}(\theta)$.

6. Determination of the structure of the fundamental group G^* of the curve $f^*_{6m}(x, y) = 0$. The component transformations of t'^2 associated with the motions corresponding to the critical values of the function $f^*_{6m}(x, y)$ give rise to relations among the generators g_k of G^* of the following types:

1°. From the transformations associated with the motions (19), we obtain the relations

$$(20.j) \qquad g_{6m-3j-2}g_{6m-3j-1}g_{6m-3j} = g_{6m-3j-1}g_{6m-3j}g_{6m-3j+1}$$

and the relations

$$(21.j) \qquad g_{6m-3j}g_{6m-3j+1}g_{6m-3j+2} = g_{6m-3j+1}g_{6m-3j+2}g_{6m-3j+3}$$

where $j = 1, 2, \cdots, 2m-1$.

2°. The transformations $t^3_{6m-3j+1}$ give the relations

$$(22.j) \qquad g_{6m-3j+1}g_{6m-3j+2}g_{6m-3j+1} = g_{6m-3j+2}g_{6m-3j+1}g_{6m-3j+2}$$

for $j = 1, 2, \cdots, 2m$.

3°. The transformations $t^{-1}_{6m-3j+2}t_{6m-3j+1}t_{6m-3j+2}$ yield the relations

$$(23.j) \qquad g_{6m-3j+1} = g_{6m-3j+3} \text{ for } j = 1, 2, \cdots, 2m).$$

In addition, the generators g_k also satisfy the trivial relation

$$(4) \qquad g_1g_2 \cdots g_n = 1.$$

If we now apply $(23.j+1)$ to $(20.j)$ we have

$$g_{6m-3j-2}g_{6m-3j-1}g_{6m-3j-2} = g_{6m-3j-1}g_{6m-3j-2}g_{6m-3j+1}$$

and, on application of $(22.j+1)$ to the left-hand member of this expression, we obtain

$$g_{6m-3j-1}g_{6m-3j-2}g_{6m-3j-1} = g_{6m-3j-1}g_{6m-3j-2}g_{6m-3j+1}$$

whence

$$(24) \qquad g_{6m-3j-1} = g_{6m-3j+1} \text{ for } j = 1, \cdots, 2m-1.$$

In the same way, if we apply $(23.j)$ to $(21.j)$ we obtain

$$g_{6m-3j}g_{6m-3j+1}g_{6m-3j+2} = g_{6m-3j+1}g_{6m-3j+2}g_{6m-3j+1}$$

and, by application of (22. j) to the right-hand member, this expression reduces to

$$g_{6m-3j}g_{6m-3j+1}g_{6m-3j+2} = g_{6m-3j+2}g_{6m-3j+1}g_{6m-3j+2}$$

whence,

(25) $$g_{6m-3j} = g_{6m-3j+2} \text{ for } j = 1, \cdots, 2m-1.$$

On combining the relations (23), (24) and (25) we obtain

(26) $$g_1 = g_3 = \cdots = g_{6m-1}; \qquad g_2 = g_4 = \cdots = g_{6m}.$$

If we make use of the relations (26), we are enabled to eliminate the generators $g_3, g_4, \cdots, g_{6m}$ from the relation (4) obtaining

(27) $$(g_1g_2)^{3m} = 1.$$

Thus, G^* may be generated by two elements g_1 and g_2 satisfying the relations (22. $2m$) and (27). The relation (22. $2m$) gives as a consequence the relation $(g_1g_2)^3 = (g_1g_2g_1)^2$ and conversely.

Let us define new elements U and V in the following manner:

(28) $$U = g_1g_2g_1; \qquad V = g_1g_2.$$

Then the relations:

(29) $$U^2 = V^3; \qquad U^{2m} = 1,$$

together with the defining relations (28), give as consequences the relations (27) and (22. $2m$). Hence, it is possible to generate the fundamental group, G^*, of the curve $f^*_{6m}(x, y) = 0$ by the two elements U and V satisfying the relations (29).

7. Conclusion. The curve $f^*_{6m} = 0$ is of the same type as $f_{6m} = 0$. Moreover, the number and kind of singularities are the same for both curves and the restrictive hypothesis to the effect that the cusps of $f_{6m}(x, y) = 0$ should be distinct is also satisfied for $f^*_{6m}(x, y) = 0$. Consequently, it is clear that the curves are isotopic and, therefore, that their fundamental groups possess the same structure. Hence, we may conclude that the fundamental group G of the curves

(3) $$f_{6m}(x, y) \equiv [\phi_{3m}(x, y)]^2 + [\psi_{2m}(x, y)]^3 = 0$$

may be generated by two elements U and V, of respective orders $2m$ and $3m$, satisfying the relations $U^2V^{-3} = U^{2m} = 1$.

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